Chapter 2

The Solution of Numerical Algebraic and Transcendental Equations

Introduction

In this chapter we shall discuss some numerical methods for solving algebraic and transcendental equations. The equation $f(x)=0$ is said to be algebraic if $f(x)$ is purely a polynomial in x. If $f(x)$ contains some other functions, namely, Trigonometric, Logarithmic, Exponential, etc., then the equation $f(x)=0$ is called a Transcendental Equation.

The equations $x^3-7x+8=0$ and $x^4+4x^3+7x^2+6x+3=0$ are **algebraic**.

The equations $3\tan 3x=3x+1,\ x-2\sin x=0$ and $e^x=4x$ are **transcendental**.

Algebraically, the real number α is called the real root (or zero of the function $f(x)$) of the equation $f(x)=0$ if and only if $f(\alpha)=0$ and geometrically the real root of an equation $f(x)=0$ is the value of x where the graph of $f(x)$ meets the x-axis in rectangular coordinate system.

We will assume that the equation

$$f(x)=0 \tag{1.1}$$

has only isolated roots, that is for each root of the equation there is a neighbourhood which does not contain any other roots of the equation.
Approximately the isolated roots of the equation (1.1) has two stages.

1. Isolating the roots that is finding the smallest possible interval (a,b) containing one and only one root of the equation (1.1).

2. Improving the values of the approximate roots to the specified degree of accuracy. Now we state a very useful theorem of mathematical analysis without proof.

Theorem (1.1):

If a function $f(x)$ assumes values of opposite sign at the end points of interval (a,b),

i.e.,
$$f(a)\,f(b)<0 \tag{1.2}$$

then the interval will contain at least one root of the equation $f(x)=0$, in other words, there will be at least one number $c\in(a,b)$ such that $f(c)=0$.Throughout our discussion in this chapter we assume that

1. $f(x)$ is continuous and continuously differentiable up to sufficient number of times.

2. $f(x)=0$ has no multiple root, that is, if c is a real root $f(x)=0$ then $f(c)=0$.

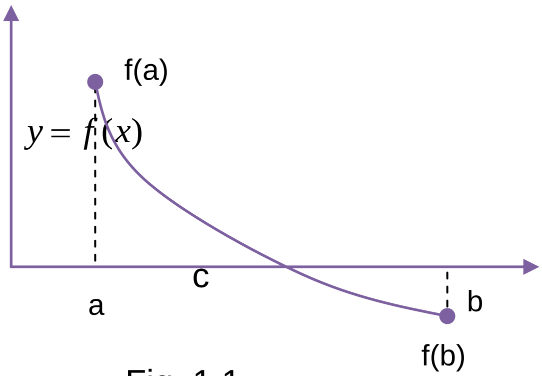


Fig. 1.1

We will discuss the following methods:

i. The bisection method
ii. The false-position method
iii. Newton-Raphson method

1.The Bisection Method

In this chapter we consider one of the most basic problems of numerical approximation، the **root-finding problem**. This process involves finding a root, or solution, of an equation of the form $f(x)=0$, for a given function f. A root of this equation is also called a zero of the function f .

BisectionTechnique

The first technique, based on the Intermediate Value Theorem, is called the **Bisection**, or **Binary-search, method**.

Suppose f is a continuous function defined on the interval $[a,b]$, with $f(a)$ and $f(b)$ of opposite sign. The Intermediate Value Theorem implies that a number c exists in (a,b) with $f(c)$. Although the procedure will work when there is more than one root in the

interval (a,b), we assume for simplicity that the root in this interval is unique. The method calls for a repeated halving (or bisecting) of subintervals of [a, b] and, at each step, locating the half containing c.

To begin, set $a_1 = a$ and $b_1 = b$, and let c_1 be the midpoint of $[a,b]$; that is,

$$c_1 = \frac{a_1 + b_1}{2}$$

- If $f(c_1)=0, c=c_1$ then, and we are done.
- If $f(c_1)\neq 0$ then $f(c_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
- If $f(c_1)$ and $f(a_1)$ have the same sign, $c\in(c_1,b_1)$. Set $a_2=c_1$ and $b_2=b_1$.
- If $f(c_1)$ and $f(a_1)$ have opposite signs, $c\in(a_1,c_1)$. Set $a_2=a_1$ and $b_2=c_1$.

Example (1): Find the largest root of $f(x)=x^6-x-1=0$ accurate to within $\varepsilon=0.001.$

Solution:

With a graph, it is easy to check that $1<\alpha<2$. We chose $a=1,b=2$; then $f(a)=-1, f(b)=61$, and $f(a)f(b)<0$. The results of the algorithm are shown in Table (1.1). The entry n indicates that the associated row corresponds to iteration number n of steps.

Table (1.1)

n	a_n	b_n	c_n	b_n-c_n	$f(c_n)$
1	1	2	1.5	0.5	8.8906
2	1	1.5	1.25	0.25	1.5647
3	1	1.25	1.125	0.125	-0.0977
4	1.125	1.25	1.1875	0.0625	0.6167
5	1.125	1.1875	1.1562	0.0312	0.2333
6	1.125	1.1562	1.1406	0.0156	0.0616
7	1.125	1.1406	1.1328	0.0078	-0.0196
8	1.1328	1.1406	1.1367	0.0039	0.0206
9	1.1328	1.1367	1.1348	0.0020	0.0004
10	1.1328	1.1348	1.1338	0.00098	-0.0096

In the 10^{th} step $b_n-c_n\le\varepsilon$. Thus $\alpha=1.33$ is the root of $f(x)=0$, accurate to within $\varepsilon=0.001.$

Error Bounds

Let a_n,b_n and c_n denote the n^{th} computed values of a,b and c, respectively. Then easily we get

$$b_1-a_1=\frac{1}{2}(b_0-a_0)$$

then

$$b_{n+1}-a_{n+1}=\frac{1}{2}(b_n-a_n), \qquad n\ge 1 \tag{1.3}$$

and it is straightforward to deduce that

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a), \qquad n \geq 1 \tag{1.4}$$

Where $b-a$ denotes the length of the original interval with which we started. Since the root α is in either the interval $[a_n, c_n]$ or $[c_n, b_n]$, we know that

$$|\alpha - c_n| \leq c_n - a_n = b_n - c_n = \frac{1}{2}(b_n - a_n) \tag{1.5}$$

This is the error bound for c_n .Combining it with (1.4), we obtain the further bounded

$$|\alpha - c_n| \leq \frac{1}{2^n}(b-a) \tag{1.6}$$

This shows that the iterations c_n converge to α as $n \to \infty$.

To see how much iteration will be necessary, suppose we want to have $|\alpha - c_n| \leq \varepsilon$

This will be satisfied if

$$\frac{1}{2^n}(b-a) \leq \varepsilon$$

Taking logarithms of both sides, we can solve this to give

$$n \geq \frac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log 2} \tag{1.7}$$

For our example (1), this results in

$$n \geq \frac{\log\left(\frac{1}{0.001}\right)}{\log 2} = 9.97$$

Thus, we must have $n = 10$ iterates, exactly the number computed.

Example (2): Compute one root of $e^x - 3x = 0$ correct to two decimal places.

Solution:

Let $f(x) = e^x - 3x = 0$ .With a graph, it is easy to check that $1.5 < \alpha < 1.6$. We chose $a = 1.5, b = 1.6$; then $f(1.5) = -0.02$, $f(1.6) = 0.15$, and $f(a)f(b) < 0$. The results of the algorithm are shown in Table (1.2). The entry n indicates that the associated row corresponds to iteration number n of steps.

Table (1.2)

n	a_n	b_n	c_n	$f(c_n)$
1	*1.5*	*1.6*	*1.55*	*0.06*
2	*1.5*	*1.55*	*1.525*	*0.02*
3	*1.5*	*1.525*	*1.5125*	*0.00056*
4	*1.5*	*1.5125*	*1.5062*	*-0.00904*
5	*1.5062*	*1.5125*	*1.50935*	*-0.00426*

In the 4^{th} step a_n, b_n and c_n are equal up two decimal places. Thus $\alpha = 1.51$ is the root of $f(x) = 0$, correct up two decimal places.

Example (3): Find the cube root of 7 using the Bisection method. Find the answer correct to 2 decimal places and work to 3 decimal places throughout.

Solution:

We need to solve $x^3 = 7$ for x. Let $f(x) = x^3 - 7$ then we need to find x where $f(x) = 0$.

If $x = 2$, $f(2) = 8 - 7 = +1$,

If $x = 1$, $f(2) = 1 - 7 = -6$,

$\Rightarrow f(x) = 0$ for some x value between 1 and 2.

$\therefore$ $[1, 2]$ is our starting interval. Iterations are best laid out in a Table (1.3)

Table (1.3)

n	a_n	b_n	c_n	$f(c_n)$
1	*1*	*2*	*1.5*	*-3.625*
2	*1.5*	*2*	*1.75*	*-1.641*
3	*1.75*	*2*	*1.875*	*-0.408*
4	*1.875*	*2*	*1.938*	*-0.279*
5	*1.875*	*1.938*	*1.907*	*-0.065*
6	*1.907*	*1.938*	*1.923*	*+0.111*
7	*1.907*	*1.923*	*1.915*	*+0.023*
8	*1.907*	*1.915*	*1.911*	*-0.021*

The last row tells us that the root lies between ***1.911*** and ***1.915*** but is not **1.915** exactly. This means that the root must be **1.91** to 2 decimal places.i.e. Cube root of 7 is **1.91** to 2 d.p.

2.False Position or Regula Falsi Method

Similarly to the bisection method, the false position or regula falsi method starts with the initial solution interval $[a,b]$ that is believed to contain the solution of $f(x) = 0$. Approximating the curve of $f(x)$ on $[a,b]$ by a straight line connecting the two points $(a, f(a))$ and $(b, f(b))$, the graph of $y = f(x)$ will meet the x-axis at the same point between a and b , the equation chord joining the two points $[a, f(a)]$ and $[b, f(b)]$ is

$$\frac{y - f(a)}{(x - a)} = \frac{f(b) - f(a)}{b - a}$$

in the small interval (a,b) the graph of the function can be considered as a straight line. So that x-coordinate of the point of intersection of the chord joining $[a, f(a)]$ and $[b, f(b)]$ with the x-axis will give an approximate value of the root. So putting $y=0$.

$$\frac{-f(a)}{(x-a)}=\frac{f(b)-f(a)}{b-a} \Rightarrow x=a+\frac{-f(a)}{f(b)-f(a)}(b-a)$$

Or

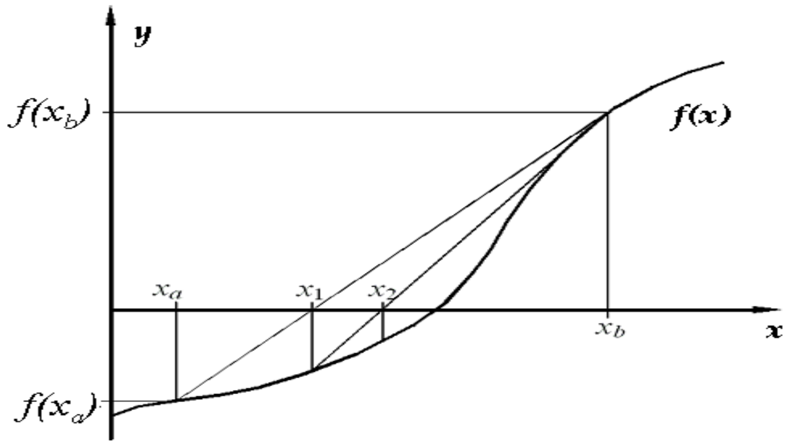
$$x=c=\frac{a\,f(b)-b\,f(a)}{f(b)-f(a)} \tag{1.8}$$

- If $f(c)=0$, $x=c$ then, and we are done.
- If $f(a)f(c)<0$, a zero lies in the interval $[a,c]$, so we set $b=c$.
- If $f(b)f(c)<0$, a zero lies in the interval $[c,b]$, so we set $a=c$.

Then we use the secant formula to find the new approximation for x^* :

$$x_2=c=\frac{a\,f(b)-b\,f(a)}{f(b)-f(a)}$$

We repeat the process until we find an x_n with $|x_n - x_{n-1}|<\varepsilon$ or $|f(x_n)|<\varepsilon$.



Example (4): Find the root between $(2,3)$ of $x^3-2x-5=0$, by using regular falsi method.

Solution:

Given $f(x)=x^3-2x-5$

$f(2)=8-4-5=-1$ (Negative)

$f(3)=27-6-5=16$ (Positive)

Let us take $a=2$ and $b=3$.

The first approximation to root is x_1 and is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2\,f(3) - 3\,f(2)}{f(3) - f(2)} = \frac{2(16) - 3(-1)}{(16) - (-1)} = \frac{35}{17} = 2.058$$

Now $f(2.058) = (2.058)^3 - 2(2.058) - 5 = 8.716 - 4.116 - 5 = -0.4$

The root lies between 2.058 and 3

Taking $a = 2.058$ and $b = 3$.

We have the second approximation to the root given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.058\,f(3) - 3\,f(2.058)}{f(3) - f(2.058)} = \frac{2.058(16) - 3(-0.4)}{(16) - (-0.4)} = 2.081$$

Now $f(2.081) = (2.081)^3 - 2(2.081) - 5 = -0.15$

The root lies between 2.081 and 3

Taking $a = 2.081$ and $b = 3$.

We have the third approximation to the root given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.081\,f(3) - 3\,f(2.081)}{f(3) - f(2.081)} = \frac{2.081(16) - 3(-0.15)}{(16) - (-0.15)} = 2.093$$

So the root is 2.09

Example (5): Find an approximate root of $x\,log_{10}x - 1.2 = 0$ by false position method

Solution:

Let $f(x) = x\,log_{10}x - 1.2$

$$f(1) = \text{-}1.2 \quad ;\; f(2) = 2(0.30103)\text{-}1.2 = \text{-}0.597940$$

$$f(3) = 3(0.47712)\text{-}1.2 = 0.231364$$

So, the root lies between 2 and 3.

Let us take $a = 2$ and $b = 3$.

The first approximation to root is x_1 and is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2\,f(3) - 3\,f(2)}{f(3) - f(2)} = \frac{2(0.231364) - 3(\text{-}0.597940)}{(0.231364) - (\text{-}0.597940)} = \frac{35}{17} = 2.721014$$

Now $f(2.721014) = \text{-}\,0.017104$

The root lies between 2.058 and 3

Taking $a = 2.721014$ and $b = 3$.

We have the second approximation to the root given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.721014\,f(3) - 3\,f(2.721014)}{f(3) - f(2.721014)}$$

$$x_2 = \frac{2.721014(0.231364) - 3(2.721014)}{(0.231364) - (2.721014)} = 2.740211$$

Now $f\left(2.740211\right) = 2.740211\left(log(2.740211)\right) - 1.2 = -0.00038905$

So the root lies between 2.740211 and 3

Taking $a = 2.740211$ and $b = 3$.

We have the third approximation to the root given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.740211\,f(3) - 3\,f(2.740211)}{f(3) - f(2.740211)}$$

$$x_3 = \frac{2.740211(0.231364) - 3(-0.00038905)}{(0.231364) - (-0.00038905)} = 2.740627$$

Now $f\left(2.740627\right) = 2.740627\left(log(2.740627)\right) - 1.2 = 0.00011998$

So the root lies between 2.740211 and $2.740627$3

Taking $a = 2.740211$ and $b = 2.740627$.

We have the fourth approximation to the root given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2.740211\,f(2.740627) - 2.740627\,f(2.740211)}{f(2.740627) - f(2.740211)}$$

$$x_4 = \frac{2.740211(0.00011998) - 2.740627(-0.00038905)}{(0.00011998) - (-0.00038905)} = 2.7405$$

Hence the root is 2.7405.

3.Newton–Raphson Method Or Newton Iteration Method

This is also an iteration method and is used to find the isolated roots of an equation $f(x) = 0$, when the derivative of $f(x)$ is a simple expression. It is derived as follows:

Suppose that $f \in C^2[a,b]$. Let $x_0 \in [a,b]$ be an approximation to x such that $f'(x_0) \neq 0$ and $\left|x - x_0\right|$ is “small.” Let $x = x_0$ be an approximate value of one root of the equation $f(x) = 0$. If $x = x_1$, is the exact root then

$$f(x_1) = 0 \qquad (1.9)$$

where the difference between x_0 and x_1 is very small and if h denotes the difference then

$$x_1 = x_0 + h \tag{1.10}$$

Substituting in (1.9) we get

$$f(x_1) = f(x_0 + h) = 0$$

Consider the first Taylor polynomial for $f(x)$ expanded about x_0.

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ldots = 0 \tag{1.11}$$

Since h is small, neglecting all the powers of h above the first from (1.11) we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{Approximately}$$

$$\Rightarrow \qquad h = -\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ from (1.9) we get

$$\Rightarrow \qquad x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{1.12}$$

The above value of x_1 is a closer approximation to the root of $f(x) = 0$ than x_0. Similarly if x_2 denotes a better approximation, starting with x_1, we get

$$x_2 = x_1 + h = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Proceeding in this way we get

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1.13}$$

The above is a general formula, known as Newton–Raphson formula.

Geometrical Derivation: The equation of the tangent line to the graph $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$f'(x_0) = \frac{y(x) - f(x_0)}{(x - x_0)} \Rightarrow y(x) - f(x_0) = (x - x_0) f'(x_0)$$

The tangent line intersects the x -axis when $y = f(x) = 0$ and $x = x_1$, so

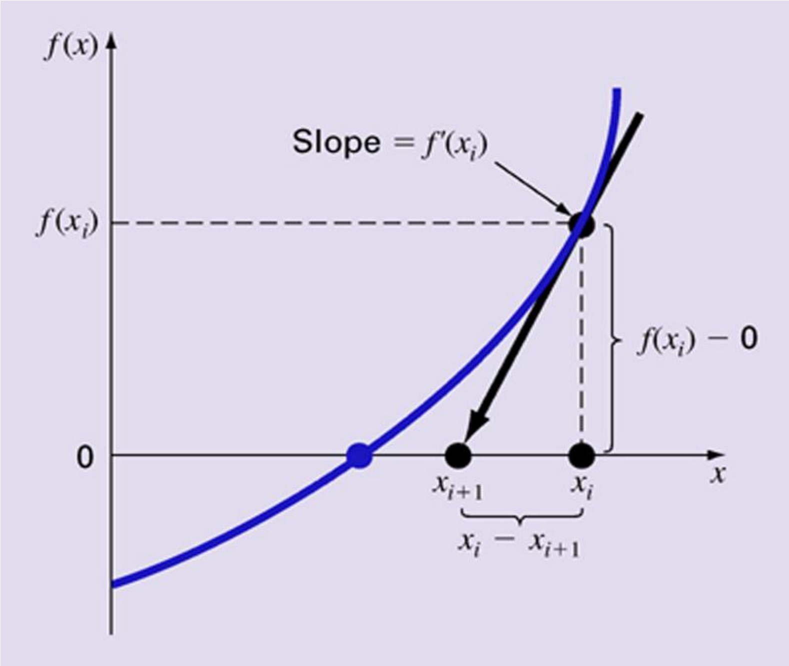
$$0 - f(x_0) = (x_1 - x_0) f'(x_0)$$

Solving this for x_1 gives

$$x_1 = x_0 + \frac{f(x_0)}{f'(x_0)}$$

and, more generally,

$$x_{i+1} = x_i + \frac{f(x_i)}{f'(x_i)} \tag{1.14}$$



Note:

Criterion for ending the iteration: The decision of stopping the iteration depends on the accuracy desired by the user. If ε denotes the tolerable error, then the process of iteration should be terminated when $|x_{i+1} - x_i| \leq \varepsilon$.In the case of linearly convergent methods the process of iteration should be terminated when $|f(x_i)| \leq \varepsilon$ where ε is tolerable error.

Example (6): Find cube root or 24, correct to three places of decimal by Newton-Raphson method.

Solution:

Finding the cube root of 24 is same as solving the equation $x^3 - 24 = 0$.

Let $f(x) = x^3 - 24 \Rightarrow f'(x) = 3x^2$

Now we have Newton-Raphson formula as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \qquad x_{n+1} = x_n - \frac{x_n^3 - 24}{3x_n^2}$$

$$\Rightarrow \qquad x_{n+1} = \frac{3x_n^3 + 24}{3x_n^2} \tag{1}$$

Now for $n=0$, $x_0=2.8$, we have

$$x_1=\frac{2(2.8)^3+24}{3(2.8)^2}=2.887.$$

For second approximation, put $n=1$, $x_1=2.887$ in (1), we get

$$x_2=\frac{2(2.887)^3+24}{3(2.887)^2}=2.885.$$

For third approximation, put $n=2$, $x_2=2.8845$ in (1), we get

$$x_3=\frac{3(2.885)^3+24}{3(2.885)^2}=2.8844.$$

Thus the cube root of 24 is 2.884.

Example (7): Apply Newton-Raphson method to find an approximate root, correct to three decimal places, of the equation $x^3-3x-5=0$,which lies near $x=2$.

Solution:

Here $f(x)=x^3-3x-5$ and $f'(x)=3x^2-3$

The Newton-Raphson iterative formula is $x_{i+1}=x_i+\frac{f(x_i)}{f'(x_i)}$

$$x_{i+1}=x_i+\frac{f(x_i)}{f'(x_i)}=x_i-\frac{x_i^3-3x_i-5}{3\left(x_i^2-1\right)}=\frac{2x_i^3+5}{3\left(x_i^2-1\right)},\ i=0,1,2,3,\ldots\ldots \qquad (1)$$

To find the root near $x=2$,we put $i=0$ in (1) and $x_0=2$ then

$$x_1=x_0-\frac{x_0^3-3x_0-5}{3\left(x_0^2-1\right)}=\frac{2x_0^3+5}{3\left(x_0^2-1\right)}=\frac{16+5}{3(4-1)}=\frac{21}{9}=2.3333$$

For second approximation, put $i=1$, $x_1=2.3333$ in (1), we get

$$x_2=x_1-\frac{x_1^3-3x_1-5}{3\left(x_1^2-1\right)}=\frac{2x_1^3+5}{3\left(x_1^2-1\right)}=\frac{2(2.3333)^3+5}{3(2.3333-1)}=2.2806$$

For third approximation, put $i=2$, $x_2=2.2806$ in (1), we get

$$x_3=x_2-\frac{x_2^3-3x_2-5}{3\left(x_2^2-1\right)}=\frac{2x_2^3+5}{3\left(x_2^2-1\right)}=\frac{2(2.2806)^3+5}{3(2.2806-1)}=2.2790$$

For the fourth approximation, put $i=3$, $x_3=2.2790$ in (1), we get

$$x_4=x_3-\frac{x_3^3-3x_3-5}{3\left(x_3^2-1\right)}=\frac{2x_3^3+5}{3\left(x_3^2-1\right)}=\frac{2(2.2790)^3+5}{3(2.2790-1)}=2.2790$$

Since x_3 and x_4 are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal.

Example (8): Find the real root of $xe^x - 2 = 0$ correct to three places of decimals using Newton-Raphson method.

Solution:

Let $f(x) = xe^x - 2$

Now, $f(0) = -2 < 0$ and $f(1) = e - 2 = 2.7183 - 2 = 0.7183 > 0$

Thus the root lies between 0 and 1.

Now, $f'(x) = xe^x + e^x = e^x(x+1)$

Now, $|f(0)| > |f(1)|$. Thus we shall take the initial approximation $x_0 = 1$. The successive approximation are shown in the following table.

n	x_n	x_{n+1}
0	$x_0 = 1$	$x_1 = 0.8679$
1	$x_1 = 0.8679$	$x_2 = 0.528$
2	$x_2 = 0.8528$	$x_3 = 0.8526$

Since x_2 and x_3 are equal up to three decimals, we have the required root as 0.853.

Exercise

1. Find the positive real root of the equation $x\log_{10} x=1.2$, using bisection method in four iteration
2. Find a real positive root of the equation $x^3-7x+5=0$ by using bisection method, correct to three places of decimal.
3. Verify that the function $f(x)=x-cos(x)$ has a root in the interval $[0,1]$ and hence apply the bisection method, in only five iterations, to approximate the root.
4. Find the real root of $3x\text{-}\cos x\text{-}1=0$ by Newton-Raphson method correct to 4 places.
5. Find the cube root of 12 applying the Newton–Raphson formula.
6. Using Newton-Raphson method establish the iterative formula $x_{n+1}=\frac{1}{3}\left[2x_n+\frac{N}{x_n^2}\right]$ to calculate the cube root of *N*.
7. Using Newton–Raphson formula, establish the iterative formula $x_{n+1}=\frac{1}{2}\left[x_n+\frac{N}{x_n}\right]$ to calculate the square root of *N*.
8. Starting with $x_0=1$ by using Newton-Raphson method approximate a root of the function $f(x)=x^2+4x-10$ correct to 6 decimal places.
9. Find an approximate value of the root of the equation $x^3+x-1=0$ near $x=1$, by the method of Falsi using the formula twice.
10. Solve the equation $x\,tan\,x=-1$, by Regula falsi method starting with 2.5 and 3.0 as the initial approximations to the root.
11. Use the false position method to find the root of $x\,sin(x)-1=0$ that is located in the interval $[0,2]$